

# Pressure and Maxwell Tensor in a Coulomb Fluid

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The pressure in a classical Coulomb fluid at equilibrium is obtained from the Maxwell tensor at some point inside the fluid, by a suitable statistical average. For fluids in a Euclidean space, this is a fresh look at known results. But for fluids in a curved space, a case which is of some interest, these unambiguous results from the Maxwell tensor approach have not been obtained by other methods.

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**KEY WORDS:** Coulomb fluids; pressure; Maxwell tensor; curved space.

## 1. INTRODUCTION

The aim of the present paper is to revisit the concept of pressure for a Coulomb fluid, i.e., a fluid made of particles interacting through Coulomb's law (electrolyte, plasma,...). We consider only fluids in thermodynamic equilibrium, and assume that classical (i.e., non-quantum) statistical mechanics is applicable.

Pressure is often defined as the force per unit area that a fluid exerts on the walls of a (large) vessel containing it. However, pressure may also be defined without reference to any wall. One has to *imagine* some *immaterial* plane surface across the fluid, and pressure is the force per unit area with which the fluid lying on one side of this surface pushes on the fluid lying on the other side. Both definitions agree with each other.

From a microscopic point of view, the force between two parts of the fluid is usually described in terms of the interactions between the molecules. Two molecules at a distance  $r$  from each other are supposed to interact through some potential. This is the standard approach, which is briefly recalled in Section 2.

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In the case of electromagnetism, Maxwell, following Faraday, came to a different point of view: the forces between two charged objects are mediated by fields. At a given point of space, even in vacuum outside the charges, there is a stress tensor (the Maxwell stress tensor), a local quantity defined in terms of the fields at that point, similar to the stress tensor within some elastic medium. In this picture, every region of "empty" space exerts forces on the regions adjacent to it. In Section 3, it is shown how the pressure at some point inside a Coulomb fluid can be defined and computed from the Maxwell tensor at that point by a suitable statistical average, with an appropriate prescription for obtaining a finite result. This result agrees with the standard one.

Section 3 "extends" the above ideas to the case of two-dimensional models.

Section 4 discusses the case of Coulomb fluids living in a curved space. In this case, the Maxwell tensor approach will be shown to be especially appropriate.

## 2. A SUMMARY OF THE STANDARD APPROACH

In the simple case of a fluid made of one species of particles, with a pair interaction  $v(r)$  depending only on the distance  $r$ , the pressure  $P$  is found to be, in the thermodynamic limit,

$$P = nkT - \frac{1}{6} n^2 \int \frac{dv}{dr} r g(r) d\mathbf{r} \quad (2.1)$$

where  $n$  is the number density (number of particles per unit volume),  $k$  is Boltzmann's constant,  $T$  is the temperature, and  $g(r)$  is the pair distribution function. In (2.1),  $nkT$  is the ideal gas part of the pressure (related to the momentum carried by the particles), while the following term, due to the interactions, is called the excess pressure  $P_{\text{ex}}$ . The same equation (2.1) is obtained by looking either at the pressure on the walls<sup>(1)</sup> or at the pressure in the bulk fluid.<sup>(2)</sup>

Real Coulomb fluids are made of several species of particles (for instance, in an electrolyte, positive and negative ions, plus the solvent molecules). In a classical model, some short-range non-Coulombic interaction must be introduced, for avoiding the collapse on each other of particles of opposite sign. Here, for simplicity, we rather consider only a simplified model, the one-component plasma (OCP):<sup>(3)</sup> identical point-particles of one sign, each of them carrying an electric charge  $q$ , embedded in a uniform background of opposite sign which ensures overall neutrality. Only the Coulomb interaction is retained, thus the interaction is  $v(r) = q^2/r$

and  $(dv/dr)r = -q^2/r$ . Due to the background, the average charge in a volume element  $d\mathbf{r}$  at a distance  $r$  of a given particle is  $qn[g(r) - 1]$  rather than  $qng(r)$ , and, in the case of the OCP, Eq. (1) is to be replaced by

$$P = nkT + \frac{q^2 n^2}{6} \int \frac{1}{r} h(r) d\mathbf{r} \quad (2.2)$$

where  $h(r) = g(r) - 1$  is the pair correlation function. It may be noted that  $P_{\text{ex}}$  is one third of the potential energy density.<sup>2</sup>

The above standard approach is based on the assumption of an interaction-at-distance  $q^2/r$ . In the next Section, it will be shown how (2.2) can be derived by using the Maxwell tensor.

### 3. THE MAXWELL TENSOR APPROACH

If only electrostatic interactions are retained (magnetic effects are neglected), the Maxwell tensor is<sup>(5)</sup>

$$T_{\alpha\beta} = \frac{1}{4\pi} \left( E_\alpha E_\beta - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \delta_{\alpha\beta} \right) \quad (3.1)$$

In (3.1), the Greek indices label the three Cartesian axes ( $x, y, z$ ).  $T_{\alpha\beta}$  is the  $\alpha$  component of the force per unit area transmitted, across a plane normal to the  $\beta$  axis, to the fluid lying on the negative side of this plane. Thus, choosing for  $\beta$  any axis, say the  $x$  axis, one obtains for the excess pressure, which is a force along that axis,

$$P_{\text{ex}} = -\langle T_{xx} \rangle = -\frac{1}{8\pi} \langle E_x^2 - E_y^2 - E_z^2 \rangle \quad (3.2)$$

where  $\langle \dots \rangle$  denotes a statistical average on all particle configurations (the electric field at some point is a function of the particle configuration). Our task is to evaluate the statistical average (3.2) at some point inside the fluid, say at the origin.

Let  $\rho^{(2)}(r_{12})$  be the statistical average of the product microscopic charge density at  $\mathbf{r}_1$  times microscopic charge density at  $\mathbf{r}_2$  ( $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ ). From (3.2),

$$P_{\text{ex}} = -\frac{1}{8\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{x_1 x_2 - y_1 y_2 - z_1 z_2}{r_1^3 r_2^3} \rho^{(2)}(r_{12}) \quad (3.3)$$

<sup>2</sup> For an OCP, there are several non-equivalent possible definitions of the pressure. The pressure (2.2) is the thermal pressure, in the sense of Choquard *et al.*<sup>(4)</sup>

In the present case of an OCP,

$$\rho^{(2)}(r_{12}) = q^2 [n\delta(\mathbf{r}_{12}) + n^2 h(r_{12})] \quad (3.4)$$

Using (3.4) in (3.3) gives to  $P_{\text{ex}}$  two contributions  $P_{\text{self}}$  and  $P_{\text{nonself}}$  involving the  $\delta$  part and the  $h$  part of (3.4), respectively.  $P_{\text{nonself}}$  gives no difficulty. This is a convergent integral (indeed,  $h$  is  $-1$  at small  $r_{12}$  because the particles strongly repel each other, and  $h$  has a fast decay at large  $r_{12}$  because remote particles are uncorrelated). Because of the rotational symmetry around the origin, it can be rewritten as

$$P_{\text{nonself}} = \frac{q^2 n^2}{24\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1^3 r_2^3} h(r_{12}) \quad (3.5)$$

But

$$P_{\text{self}} = -\frac{nq^2}{8\pi} \int d\mathbf{r} \frac{x^2 - y^2 - z^2}{r^6} \quad (3.6)$$

diverges at small  $r$ .

The resolution of the difficulty is that the force that each particle exerts on itself should not be taken into account. Thus, the integral in (3.6) must be regularized by the prescription that no particle sits on the  $x=0$  plane on which we have chosen to compute the pressure force. This prescription can be enforced by removing from the integration domain a thin slab  $-\varepsilon < x < \varepsilon$  and taking the limit  $\varepsilon \rightarrow 0$  at the end. This prescription does not change the convergent integral (3.5). But it means that the self part (3.6) must be defined, in cylindrical coordinates  $(x, \rho)$ , as

$$P_{\text{self}} = -\frac{nq^2}{8\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x| > \varepsilon} dx \int_0^\infty 2\pi d\rho \rho \frac{x^2 - \rho^2}{(x^2 + \rho^2)^3} \quad (3.7)$$

Since the integral on  $\rho$ , performed first, is found to vanish, the result is  $P_{\text{self}} = 0$ .

As to  $P_{\text{nonself}}$ , (3.5) can be easily computed by taking as integration variables  $\mathbf{r}_1$  and  $\mathbf{r}_{12}$ , and performing the integral on  $\mathbf{r}_1$  first with the result  $4\pi/r_{12}$ . The final result is

$$P_{\text{ex}} = P_{\text{nonself}} = \frac{q^2 n^2}{6} \int d\mathbf{r}_{12} \frac{1}{r_{12}} h(r_{12}) \quad (3.8)$$

in agreement with the standard formula (2.2).

An alternative way of calculating  $P_{\text{self}}$  will turn out to be more appropriate for extensions which follow. (3.6) is split into the contributions  $P_0$  of  $r < r_0$  and  $P_1$  of  $r > r_0$ , where  $r_0$  is the radius of a small sphere centered at the origin. The prescription that no particle sits on the plane  $x = 0$  does not change the convergent part  $P_1$ , which can be computed, using the rotational symmetry, as

$$P_1 = \frac{nq^2}{24\pi} \int_{r > r_0} \frac{d\mathbf{r}}{r^4} = \frac{nq^2}{6r_0} \tag{3.9}$$

It is only in  $P_0$  that the rotational symmetry is broken by the prescription  $|x| > \varepsilon$ , which gives

$$P_0 = -\frac{nq^2}{8\pi} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x| < r_0} dx \int_0^{\sqrt{r_0^2 - x^2}} 2\pi d\rho \rho \frac{x^2 - \rho^2}{(x^2 + \rho^2)^3} = -\frac{nq^2}{6r_0} \tag{3.10}$$

Thus  $P_{\text{self}} = P_0 + P_1 = 0$ , and (2.2) is retrieved.

### 4. TWO-DIMENSIONAL MODELS

Two-dimensional models of Coulomb fluids are of interest for at least two reasons. First, some of these models are physically relevant. Second, exact results are available. The two-dimensional case has special features which require the present separate discussion.

In two dimensions, the Coulomb interaction (as defined through the Poisson equation) between two charges  $q$  and  $q'$  is  $-qq' \ln(r/L)$ , where  $L$  is some irrelevant length. Since this interaction diverges at  $r = 0$  only mildly, in addition to the OCP it is also possible to consider a two-component plasma (TCP), made of positive and negative point-particles of respective charges  $q$  and  $-q$ , without any additional short-range repulsion (which is stable provided that the coupling constant  $\Gamma = q^2/kT$  be smaller than 2).

For the OCP, the two-dimensional analog of (2.1), with the background taken into account, is

$$P = nkT - \frac{1}{4} n^2 \int \frac{dv}{dr} rh(r) d\mathbf{r} \tag{4.1}$$

Now  $v(r) = -q^2 \ln(r/L)$  and (4.1) becomes

$$P = nkT + \frac{1}{4} n^2 q^2 \int h(r) d\mathbf{r} \tag{4.2}$$

Perfect screening, present in a conductor, says that

$$n \int h(r) d\mathbf{r} = -1 \quad (4.3)$$

(this means that the polarization cloud around a particle of charge  $q$  carries the opposite charge  $-q$ ). Using (4.3) in (4.2) gives the simple exact equation of state<sup>(6, 7)</sup>

$$P = n \left( kT - \frac{q^2}{4} \right) \quad (4.4)$$

Now,  $P_{\text{ex}} = -nq^2/4$  is no longer related to the potential energy density.

We now turn to the Maxwell tensor approach. In two dimensions, the Maxwell tensor is

$$T_{\alpha\beta} = \frac{1}{2\pi} \left( E_\alpha E_\beta - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \delta_{\alpha\beta} \right) \quad (4.5)$$

with Greek indices now labeling two Cartesian axes ( $x, y$ ). (3.3) is replaced by

$$P_{\text{ex}} = -\frac{1}{4\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{x_1 x_2 - y_1 y_2}{r_1^2 r_2^2} \rho^{(2)}(r_{12}) \quad (4.6)$$

where, for an OCP, (3.4) still holds. Now, although (4.6) still converges for large values of  $r_1$  and  $r_2$  (because  $\rho^{(2)}(r_{12})$  has a fast decay as  $r_{12}$  increases and its integral vanishes), separating it in self and nonself parts would generate terms separately diverging at infinity. Here, it is more appropriate to split (4.6) in another way, similar to what has been done at the end of Section 3. Namely, in (4.6), one separates the contribution  $P_0$  of the integration domain ( $r_1, r_2 < r_0$ ) and the rest  $P_{\text{ex}} - P_0$ . This rest is a convergent integral and, by rotational symmetry, it vanishes. One is left with  $P_0$  which can be split into its self and nonself parts, with now a nonself part which is convergent and also vanishes by rotational symmetry. Finally, the self part has to be defined in the same way as (3.10), and

$$P_0 = -\frac{nq^2}{4\pi} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x| < r_0} dx \int_{-\sqrt{r_0^2 - x^2}}^{\sqrt{r_0^2 - x^2}} dy \frac{x^2 - y^2}{(x^2 + y^2)^2} = -\frac{nq^2}{4} \quad (4.7)$$

Thus

$$P_{\text{ex}} = P_0 = -\frac{nq^2}{4} \quad (4.8)$$

in agreement with (4.4).

Similar considerations hold for the TCP, as long as  $\Gamma < 2$ , and the equation of state again is (4.4), where now  $n$  is the total number density of the particles.

The two-dimensional OCP can also be obtained as a limit of the  $\nu$ -dimensional one, as explained in Appendix A.

## 5. CURVED SPACES

The statistical mechanics of a Coulomb fluid living in a curved space is of interest for at least two reasons. First, for doing numerical simulations (necessarily on a finite system) without having to deal with boundary effects, a clever method has been to confine the system on the surface of a sphere (in the two-dimensional case)<sup>(8)</sup> or an hypersphere (in the three-dimensional case).<sup>(9, 10)</sup> Second, for two-dimensional Coulomb fluids on a surface of constant negative curvature (pseudosphere),<sup>(11)</sup> it is possible to go to the limit of an infinite system while keeping a finite curvature, thus to look at the properties of a curved infinite system (something which cannot be done for a sphere or hypersphere).

The present paper actually arose from the question: How to define the pressure of a Coulomb fluid in a curved space, away from any wall? A formula like (2.1) is based on the interaction-at-distance picture: the force acting on the fluid lying on one side of some immaterial plane is the sum of elementary forces acting on each molecule. This picture cannot be generalized to the case of a curved space, because there is no straightforward way of summing forces (vectors) applied at different points of space. Thus, the Maxwell tensor picture seems to be the only possible one, defining the pressure at a given point of space as a local quantity depending only on the electric field at this point.

Three kinds of Coulomb fluids will be considered: The three-dimensional OCP on a hypersphere, the two-dimensional OCP or TCP on a sphere, the two-dimensional OCP or TCP on a pseudosphere.

### 5.1. OCP on a Hypersphere

The hypersphere is the four-dimensional analog of the usual sphere. We consider an OCP living on the three-dimensional "surface"  $S_3$  of a hypersphere of radius  $R$ . On  $S_3$ , the geodesic distance between two points is  $R\psi$ , where  $\psi \in [0, \pi]$  is the angular distance between these points, as seen from the center of the hypersphere. The volume element between two concentric hyperspheres of radii  $R\psi$  and  $R(\psi + d\psi)$  is  $dV = 4\pi R^3 \sin^2 \psi d\psi$ . The total volume of  $S_3$  is  $V = 2\pi^2 R^3$ .

Since  $S_3$  is a compact manifold without boundary, electric potentials and fields can be defined only if the total charge is zero. In particular, the electric field created by one point charge cannot be defined. For overcoming this difficulty, one can consider the OCP as a collection of pseudo-charges:<sup>(9)</sup> a pseudocharge is defined as a point charge  $q$  plus a uniform background of total charge  $-q$ . At a point  $M$  located at a geodesic distance  $R\psi$  from a pseudocharge located at  $M_0$ , the electric potential created by the pseudocharge is

$$\Phi = \frac{q}{\pi R} \left( (\pi - \psi) \operatorname{ctn} \psi - \frac{1}{2} \right) + V_0 \quad (5.1)$$

where  $V_0$  is an arbitrary constant. The corresponding electric field at  $M$  is

$$\mathbf{E} = \frac{q}{\pi R^2} \left( \operatorname{ctn} \psi + \frac{\pi - \psi}{\sin^2 \psi} \right) \mathbf{t} \quad (5.2)$$

where  $\mathbf{t}$  is the unit vector tangent to the geodesic  $MM_0$  at  $M$ . From (5.1), the interaction energy between two pseudocharges  $i$  and  $j$  at a geodesic distance  $R\psi_{ij}$  of each other is found to be

$$\phi(\psi_{ij}) = \frac{q^2}{\pi R} \left( (\pi - \psi_{ij}) \operatorname{ctn} \psi_{ij} - \frac{1}{2} \right) \quad (5.3)$$

independent of  $V_0$ .

The excess pressure is given by (3.2) where the electric field can be written as  $\mathbf{E} = \sum \mathbf{E}_i$ , with  $\mathbf{E}_i$  the field created by the  $i$ th pseudocharge. As above, (3.2) can be split into a self part  $P_{\text{self}}$  (made of  $\mathbf{E}_i \mathbf{E}_i$  terms) and a nonself part  $P_{\text{nonself}}$  (made of  $\mathbf{E}_i \mathbf{E}_j$  ( $i \neq j$ ) terms).

Because of the rotational symmetry,  $P_{\text{nonself}}$  can be written as

$$P_{\text{nonself}} = \frac{1}{24\pi} \left\langle \sum_{i \neq j} \mathbf{E}_i \cdot \mathbf{E}_j \right\rangle = \frac{1}{3} u_{\text{nonself}} \quad (5.4)$$

where  $u_{\text{nonself}}$  is the nonself part of the potential energy density, which can be reexpressed in terms of the interaction  $\phi$  rather than in terms of fields, by the usual integration by parts, as

$$u_{\text{nonself}} = \frac{1}{2\pi^2 R^3} \left\langle \sum_{i < j} \phi(\psi_{ij}) \right\rangle = \frac{n^2}{2} \int \phi(\psi) h(\psi) dV \quad (5.5)$$

where one can use the pair correlation function  $h$  rather than the pair distribution function  $g = h + 1$  since  $\int \phi dV = 0$ .



As to  $P_{\text{self}}$ , it is a divergent integral which however can be made finite by adapting what has been done at the end of Section 3, namely splitting it into the contribution  $P_0$  of geodesic distances  $R\psi < R\psi_0$ , and the finite contribution  $P_1$  of  $R\psi > R\psi_0$  for which rotational symmetry can be used. In terms of  $E(\psi)$  given by (5.2),

$$P_1 = \frac{n}{24\pi} \int_{\psi_0}^{\pi} E^2(\psi) 4\pi R^3 \sin^2 \psi \, d\psi \tag{5.6}$$

Evaluating the integral in (5.6) as  $\psi_0 \rightarrow 0$  gives

$$P_1 = \frac{nq^2}{6R} \left( \frac{1}{\psi_0} - \frac{3}{2\pi} + O(\psi_0) \right) \tag{5.7}$$

On the other hand, for  $P_0$ , the curvature effects become negligible as  $\psi_0 \rightarrow 0$  (more precisely, as shown in Appendix B, they are  $O(\psi_0)$ ) and the Euclidean prescription (3.10) can be used, with  $r_0 = R\psi_0$ , giving

$$P_0 = -\frac{nq^2}{6R\psi_0} + O(\psi_0) \tag{5.8}$$

Finally, the total pressure is

$$P = nkT + \frac{n^2}{6} \int \phi(\psi) h(\psi) \, dV - \frac{nq^2}{4\pi R} \tag{5.9}$$

This is the generalization of (2.2) to the case of a hypersphere.

The present evaluation of the pressure makes no explicit use of the self-energy of a pseudoparticle. This is an important remark. Indeed, another possible definition of the pressure would be minus the derivative of the free energy with respect to the volume. But, for evaluating the free energy, it is necessary to define properly the zero of energy for a system of pseudoparticles, and this necessarily involves some heuristic convention about the self-energy of a pseudoparticle. In ref. 9 a reasonable convention gave  $-3q^2/4\pi R$  for this self-energy, and the corresponding pressure is identical with (5.9). This pressure does obey the usual relation

$$P_{\text{ex}} = \frac{1}{3}u \tag{5.10}$$

where  $u$  is the total potential energy density defined with the above convention. However, in ref. 10, another reasonable convention (giving a faster approach to the thermodynamic limit as  $R \rightarrow \infty$ ) has been used, with additional terms of order higher than  $1/R$ , and the pressure derived from the

corresponding free energy no longer agrees with (5.9). The definition of the pressure in terms of the Maxwell tensor is free from this arbitrariness.

## 5.2. OCP or TCP on a Sphere

The above considerations can be easily adapted to the (simpler) case of two-dimensional Coulomb systems living on the surface of a sphere. Now, the electric potential created by a pseudocharge is

$$\Phi = -q \ln \sin(\psi/2) + V_0 \quad (5.11)$$

$P_{\text{nonself}}$  and  $P_1$ , convergent integrals, vanish because of the rotational symmetry. One is left with  $P_0$ , for which the curvature effects are negligible and (4.7) holds with the same result (4.8) as in the case of a plane system. This result  $P_{\text{ex}} = -nq^2/4$  holds for an OCP, and also for a TCP when  $\Gamma < 2$ .

Here too, the free energy depends on an arbitrary convention about the zero of energy. In ref. 8, this convention was implicitly made by the way in which (5.11) was used together with the choice  $V_0 = -q \ln(2R/L)$ , with  $R$  the radius of the sphere. It is only thanks to this convention that the corresponding free energy has a derivative with respect to the sphere area which correctly gives the equation of state (4.4).

## 5.3. OCP or TCP on a Pseudosphere

Recently, two-dimensional Coulomb systems living on a surface of constant negative curvature (a pseudosphere) were studied.<sup>(11)</sup> Since, unlike a sphere, a pseudosphere is infinite, one has the interesting possibility of considering systems which are both infinite and curved.

Let  $a$  be a length such that the Gaussian curvature of the pseudosphere is  $-1/a^2$  (instead of  $1/R^2$  on a sphere). Now, the electric potential and field created by a single point charge  $q$  exist. At a geodesic distance  $s$  from this charge, the electric potential is

$$\Phi = -q \ln \tanh \frac{s}{2a} \quad (5.12)$$

where the possible additive constant has been fixed by the condition that this potential vanishes at infinity ( $s \rightarrow \infty$ ). The electric field is

$$\mathbf{E} = \frac{q}{a \sinh(s/a)} \mathbf{t} \quad (5.13)$$

where  $\mathbf{t}$  is the unit vector tangent to the geodesic.

The pressure can be obtained from the Maxwell tensor just as in the case of a sphere, with the same result (4.4), i.e.,  $P_{\text{ex}} = -nq^2/4$ . This pressure holds not only in the thermodynamic limit, but also at the center of a finite disk.

The above result for the pressure calls for some discussion. On a pseudosphere, when the size of a large domain increases, its perimeter grows as fast as its area. As a consequence, there is no uniquely defined thermodynamic limit for the free energy per particle (this limit depends on the shape of the domain and on the boundary conditions). A bulk pressure cannot be defined by deriving the free energy with respect to the area. In ref. 11, a bulk “pressure”  $p$  was defined by its virial expansion with the prescription that the thermodynamic limit of each virial coefficient  $B_k$  (which seems to exist on a pseudosphere) has to be computed before the virial series in powers of the density  $n$  is summed. It is now apparent that this  $p$  is not identical with the pressure  $P$  obtained from the Maxwell tensor in the form (4.4). We now believe that the correct pressure is  $P$ , while  $p$  only is a mathematical quantity (seemingly with some interesting properties).

Nevertheless an important result of ref. 11 is true: there is at least one thermodynamic quantity, the bulk energy per particle, which has a series expansion in integer powers of the density, in contrast to the case of a plane system in which the energy per particle is singular at zero density.

#### 5.4. Why No Trace Anomaly?

Some time ago, it has been remarked that conducting Coulomb systems are critical-like at any temperature<sup>(12, 13)</sup> in some sense: they have long-range *electric potential and field* correlations, and the free energy of a two-dimensional Coulomb system with an electric potential  $\phi$  has logarithmic finite-size corrections similar to the ones which occur<sup>(14)</sup> in a critical system described by a conformal-invariant field theory with a field  $\phi$ . In ref. 14, it was shown that, for a critical system, this logarithmic correction to the free energy is associated to a trace anomaly of the stress tensor, proportional to the Gaussian curvature of the surface on which the system lives. Thus, at first sight, one might expect that the pressure of a two-dimensional Coulomb system (which is minus one half of the expectation value of the trace of the Maxwell tensor, with a suitably defined self part) would have a term  $O(1/R^2)$  on a sphere and  $O(1/a^2)$  on a pseudosphere. Yet, such terms are not present in (4.4). Why?

Actually, for a Coulomb system made of  $N$  particles on a sphere of radius  $R$ , thus with a density  $n = N/4\pi R^2$ , the free energy  $F$  has a term

$(kT/6) \ln N$ . The pressure is (with a suitable definition of the zero of energy) a partial derivative at constant  $N$ :  $P = -(\partial F/\partial(4\pi R^2))_N$ . Thus the  $\ln N$  term in  $F$  gives no contribution to the pressure, in agreement with (4.4). However, in field theory, some ultraviolet cutoff (a length  $\eta$ ) has to be introduced and the trace of the stress tensor has an expectation value  $\langle \Theta \rangle$  related to  $(\partial F/\partial(4\pi R^2))_\eta$  with now a derivative taken at constant cutoff. When the Coulomb system is described in terms of a field theory, the role of the cutoff is played by the microscopic scale  $\eta = n^{-1/2}$ . Thus, the  $\ln N = \ln(n4\pi R^2)$  term in  $F$  is associated to a trace anomaly in the field-theoretical  $\langle \Theta \rangle$ , not in  $P$ .

A related statement is: If the expectation value of the trace of the Maxwell tensor is computed with a field-theoretical measure (the functional integral measure  $\mathcal{D}\phi$ ), it has a trace anomaly. This trace anomaly is not present when the measure is the particle configuration space one  $d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_N$ .

Similarly, the pressure (4.8) on a pseudosphere has no trace anomaly.

## 6. CONCLUSION

The pressure in a Coulomb fluid has been defined as minus the statistical average of a diagonal element, say  $T_{xx}$  of the Maxwell tensor. This definition leads to an ill-defined integral, which however can be given a definite value by an appropriate prescription: the fluid is supposed split into two regions separated by a thin empty slab normal to the  $x$ -axis,  $T_{xx}$  is computed at a point inside this slab, and the limit of a slab of zero thickness is taken at the end.

For Coulomb fluids in an Euclidean space, this approach through the Maxwell tensor is just a fresh look at well-known results. But, for Coulomb systems in a curved space, we are not aware of any other way of obtaining an unambiguous value for the pressure.

For simplicity, only point-particle systems without short-range forces have been considered. But an extension to systems with hard cores seems feasible.

## APPENDIX A. THE $\nu$ -DIMENSIONAL OCP

In this Appendix, the excess pressure of a  $\nu$ -dimensional OCP ( $\nu > 2$ ) is related to its potential energy density. The dimension  $\nu$  is treated as a continuous variable, and the limit  $\nu \rightarrow 2$  is taken.

The unit of charge is defined such that the electric field at a distance  $r$  from a unit charge be  $1/r^{v-1}$ . Thus, the potential is  $1/(v-2)r^{v-2}$ . The Maxwell tensor is

$$T_{\alpha\beta} = \frac{1}{S_{v-1}} \left( E_\alpha E_\beta - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \delta_{\alpha\beta} \right) \tag{A.1}$$

where

$$S_{v-1} = \frac{2\pi^{v/2}}{\Gamma(v/2)} \tag{A.2}$$

is the area of the sphere of unit radius.

In terms of the Maxwell tensor  $T$ , the nonself part of the pressure is

$$P_{\text{nonself}} = -\frac{1}{v} \langle \text{tr } T \rangle_{\text{nonself}} = \frac{v-2}{v} \frac{1}{2S_{v-1}} \langle \mathbf{E}^2 \rangle_{\text{nonself}} \tag{A.3}$$

where the nonself part of the electrostatic energy density is

$$\frac{1}{2S_{v-1}} \langle \mathbf{E}^2 \rangle_{\text{nonself}} = \frac{q^2 n^2}{2S_{v-1}} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1^v r_2^v} h(r_{12}) \tag{A.4}$$

Taking as integration variables  $\mathbf{r}_1$  and  $\mathbf{r}_{12}$ , and performing first the integral on  $\mathbf{r}_1$ , one finds, as expected, the potential energy density

$$\frac{1}{2S_{v-1}} \langle \mathbf{E}^2 \rangle_{\text{nonself}} = \frac{q^2 n^2}{2} \int d\mathbf{r}_{12} \frac{1}{(v-2)r_{12}^{v-2}} h(r_{12}) \tag{A.5}$$

As to the self part of the pressure  $P_{\text{self}} = -\langle T_{xx} \rangle_{\text{self}}$ , it must be defined, like in (3.7), as

$$P_{\text{self}} = -\frac{nq^2}{2S_{v-1}} \lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} dx \int_0^\infty S_{v-2} d\rho \rho^{v-2} \frac{x^2 - \rho^2}{(x^2 + \rho^2)^v} \tag{A.6}$$

Here too, the integral on  $\rho$ , performed first, is found to vanish, thus  $P_{\text{self}} = 0$ .

Therefore, the final result for the excess pressure is

$$P_{\text{ex}} = \frac{q^2 n^2}{2v} \int d\mathbf{r}_{12} \frac{1}{r_{12}^{v-2}} h(r_{12}) \tag{A.7}$$

In the limit  $\nu \rightarrow 2$ , using the perfect screening rule (4.3), one retrieves

$$P_{\text{ex}} = -\frac{nq^2}{4} \quad (\text{A.8})$$

## APPENDIX B. CURVATURE EFFECTS IN A SMALL SPHERE

In this Appendix, (5.8) is derived.

In four-dimensional Euclidean space, with Cartesian coordinates  $(x, y, z, t)$ , the surface  $S_3$  of a hypersphere of radius  $R$  centered at the origin usually is parametrized by the hyperspherical coordinates  $(u, v, w)$  related to the Cartesian ones by

$$\begin{aligned} x &= \sin w \sin v \cos u, & y &= \sin w \sin v \sin u, & z &= \sin w \cos v, & t &= \cos w \\ 0 &\leq u \leq 2\pi, & 0 &\leq v \leq \pi, & 0 &\leq w \leq \pi \end{aligned} \quad (\text{B.1})$$

However, here, it is more convenient to parametrize  $S_3$  by the three independent variables  $(x, y, z)$ . We define  $r = (x^2 + y^2 + z^2)^{1/2}$  and  $\rho = (y^2 + z^2)^{1/2}$ . A useful relation is  $r^2 = R^2 \sin^2 w$ . Using the Jacobian for the change of coordinates, one finds that the volume element  $dV = R^3 \sin^2 w \sin v \, du \, dv \, dw$  becomes

$$dV = \frac{dx \, dy \, dz}{\cos w} \quad (\text{B.2})$$

The hypersphere pole  $x = y = z = 0$  will be called  $O$ . The geodesic distance between  $O$  and  $(x, y, z)$  is  $Rw$ .

The part  $P_0$  of the pressure can be evaluated at  $O$ , i.e., the electric field in (3.2) is the one at  $O$ .  $P_0$  is that part of  $P_{\text{self}}$  which is created by the pseudocharges located at a geodesic distance from  $O$  smaller than  $R\psi_0$ . The regularization prescription is that there is no particle in a thin slab  $|x| < \varepsilon$ . The electric field  $E(w) \mathbf{t}$  created at  $O$  by a pseudocharge at  $(x, y, z)$  is given by (5.2) where  $\psi = w$  and  $\mathbf{t} = (-x/r, -y/r, -z/r)$ . Thus, with (B.2) taken into account, the analog of (3.10) is

$$P_0 = -\frac{n}{8\pi} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x| < r_0} \int_0^{\sqrt{r_0^2 - x^2}} \frac{2\pi d\rho \rho}{\cos w} \frac{x^2 - \rho^2}{x^2 + \rho^2} E^2(w) \quad (\text{B.3})$$

where  $r_0 = R \sin \psi_0$ . An expansion in powers of  $r/R$  gives

$$\frac{E^2(w)}{\cos w} = \frac{q^2}{(x^2 + \rho^2)^2} [1 + O(r^2/R^2)] \quad (\text{B.4})$$

When the expansion (B.4) is used in (B.3), the leading term of (B.3) is (3.10) and the next term (which gives a convergent integral for which the  $\varepsilon$  regularization is superfluous) is  $O(r_0/R^2)$ . Using  $r_0 = R \sin \psi_0$ , one does find

$$P_0 = -\frac{nq^2}{6R\psi_0} + O(\psi_0)$$

i.e., (5.8).

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